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Amplitude-dependent behaviour of a liquid-filled gyroscope

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An analysis of the experimentally determined resonant amplitude and frequencies of oscillation of a gyroscope whose rotor contains liquid-filled cylindrical cavities reveals a surprising small amplitude instability region that is not predicted by the Stewartson instability criterion for a liquid-filled spinning top. Along with the experimental results and speculation concerning the cause of the observed phenomenon, there are presented some practical implications of the observations of use to designers of spinning vehicles containing liquid.

1. Introduction

Experiments with a liquid-filled gyroscope by Karpov (1965) verify Stewartson's (1959) linear stability criterion for a liquid-filled top. However, more recent experiments conducted at the Ballistics Research Laboratory with a liquid-filled gyroscope indicate that the Stewartson analysis can predict the amplification rate of the gyroscope correctly only if the amplitude of motion does not exceed about one degree. For gyroscopic motion with an amplitude exceeding that value, the Stewartson linear theory not only does not predict the observed amplification rate, but also may fail completely to predict the observed instability. The instability of the gyroscope that is observed at the larger amplitudes seems to be amplitude dependent.

2. The gyroscope and the experiments

The gyroscope with its liquid-filled rotor and auxiliary equipment is shown in figure 1 (plate 1). The centre of gravity of the device is maintained at the gimbal axis in order to suppress the precessional (slow) mode of oscillation; hence, the gyroscope is really a gyrostat. The resulting circular motion is defined as nutation, the frequency of which we label τ_{nu} after non-dimensionalization with respect to the constant spin Ω of the rotor. For the gyrostat, then, τ_{nu} is simply the ratio of its axial moment of inertia to its transverse moment of inertia. τ_{nj}, a dimensionless eigenfrequency of the inertial wave motion in the uniformly spinning, inviscid, incompressible liquid in the rotor cavity, depends, as Stewartson showed,

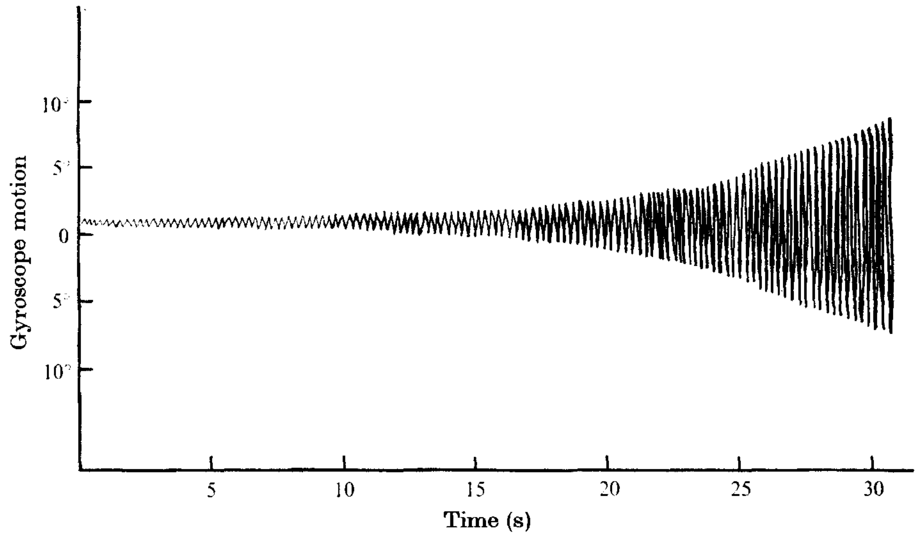

FIGURE 2. Representative plot of amplitude of motion of gyroscope in degrees versus time in seconds at resonance. Cavity height/cavity radius = 3·09, kinematic viscosity = 10 centistokes, angular speed = 6000 r.p.m., cavity completely filled.

only on the percentage of liquid in the cavity and the ratio of the cavity height $2c$ to the cavity diameter $2a$. Stewartson also showed that when the system is sufficiently close to a state of resonance, i.e. when τ_{nu} is sufficiently close to τ_{nj}, the gyrostat will become unstable, and when $\tau_{nj} = \tau_{nu}$ the amplification rate of the gyrostat will be a maximum. Hence, our experimental procedure for determining any particular eigenfrequency of the liquid involves varying the nutational frequency of the gyroscope until its amplification rate is a maximum (the subsequent correct inference that the nutational frequency of the gyrostat is then also an inertial wave frequency has been profusely documented elsewhere). This variation in τ_{nu} is accomplished by varying the moments of inertia of the rotor by adding brass rings of various masses (figure 1).

The gyrostat, which differs radically from the top used by Stewartson, is driven by a $\frac{1}{8}$ h.p. d.c. motor, and is unique in that its gimbals have flexure pivots (figure 1, lower left-hand corner) that consist of circular, crossed, spring leaves, so constructed that while half of the unit rotates about a central axis the other half remains motionless. The transverse displacement and hysteresis associated with this motion are negligible. Measurements of angular displacements of the axis of the gyrostat are made by strain gauges cemented to one of the pivot spring leaves, these strain gauges forming part of a bridge circuit whose output is amplified and continuously plotted as a function of time as shown in figure 2.

Since the natural frequencies of the liquid are functions of the percentage of liquid in the cavity, one can also search for maximum amplification (and hence resonance, and hence an eigenfrequency of the liquid) by keeping the nutational frequency of the gyrostat constant while varying the amount of liquid in the cavity. Since cavities of various dimensions can be placed in the rotor of the gyrostat, one can generate an infinity of response curves like that shown in figure 2.

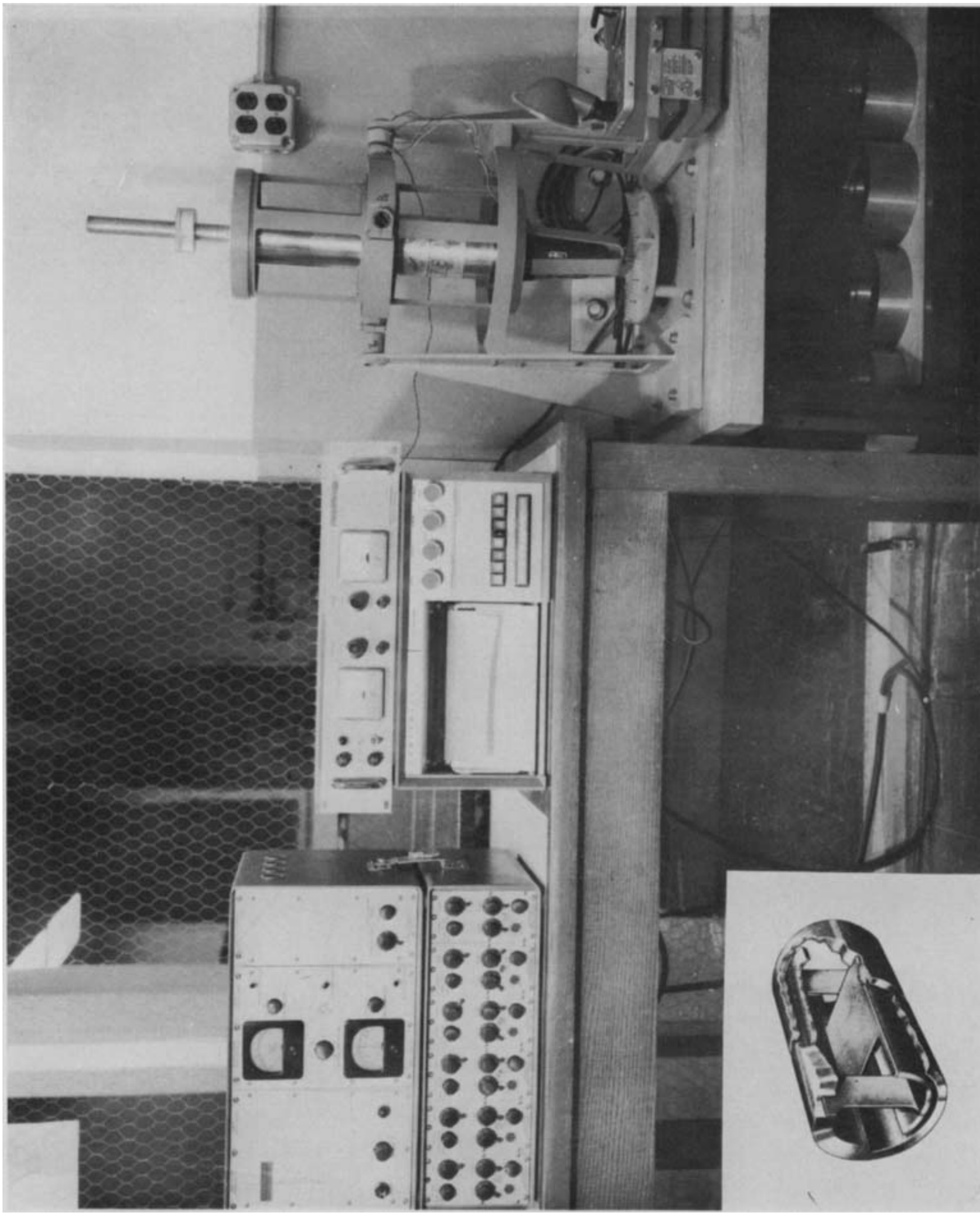

FIGURE 1. Gyrostat and supporting apparatus.

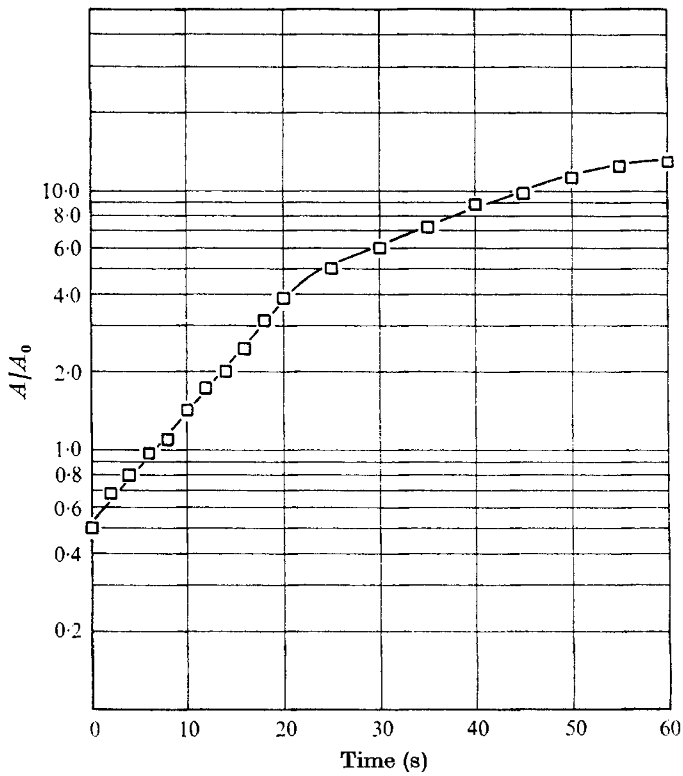

FIGURE 3. Semi-log plot of normalized amplitude of motion of gyrostat at resonance versus time in seconds. Cavity height = 7·817 in., cavity diameter = 2·50 in., angular speed = 3000 r.p.m., cavity completely filled.

One obtains an amplitude growth rate from these plots by measuring the amplitude A (normalized with respect to a convenient initial amplitude of the gyrostat A_0) at several different instants and then plotting these amplitudes as functions of time on semi-log paper. If the points lie on a straight line,† the slope of this straight line is the linear amplitude growth rate. An example of the result of this process is shown in figure 3. Note there, however, that the points lie on a straight line only for a time of about 20 s: afterwards, the growth rate seems to be nonlinearly time dependent, a fact that one easily translates into the growth rate being amplitude dependent. This latter amplitude dependence of the growth rate is not predicted by the Stewartson analysis.

Figure 4 shows a typical resonance curve for our gyrostat at very small amplitudes, e.g. about 1° (and hence characterized by the Stewartson amplitude-independent amplification or growth rate). The theoretical curve was plotted by using Stewartson's (1959) tables; the experimental curve was obtained by varying τ_{nu} while keeping the percentage of liquid and the cavity geometry fixed. The approximately 2 % difference between the abscissae of the peaks of the two

† This is so if the amplitude grows as exp (αt), where α is the amplification or growth rate and t is time.

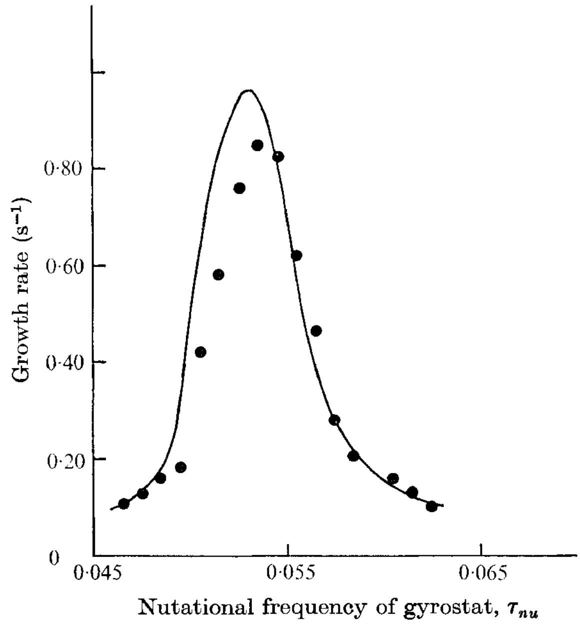

FIGURE 4. Resonance curve: amplitude growth rate versus dimensionless nutational frequency. Cavity height = 7·48 in., cavity diameter = 2·48 in., angular speed = 5000 r.p.m., cavity completely filled, dimensionless inertial wave frequency = 0·053. ●, gyrostat; ——, Stewartson's theory.

curves in the figure is an example of the excellent agreement between the predictions of the Stewartson analysis and our experiments. The agreement is even more remarkable in view of the facts that the variation obtainable in τ_{nu} was only in discrete steps of 0·001 (around 2 % of the resonance value–we did not have an infinite set of brass rings!), and that the level of amplification changes very rapidly in the vicinity of $\tau_{nu} = \tau_{nj}$ for small changes in τ_{nu}, i.e. the resonance bandwidth here is very narrow. Indeed, Stewartson's tables show how critically τ_{nj} depends on the ratio of height to diameter for our particular cavity, a 1 % change in diameter producing a 10 % change in τ_{nj} (if τ_{nj} were greater than 0·25, its sensitivity to the height-to-diameter ratio would be less, but our gyrostat was not capable of operating at such high frequencies). Hence, the experiments had to be performed with extreme care.

We now consider in more detail the amplitude-dependent behaviour depicted in figure 3. Since we used both partially and completely filled cavities, the procedure for obtaining resonance involved successively incrementing τ_{nu} and then running the gyrostat at large amplitudes to determine the large amplitude growth rate. To determine whether the large amplitude growth rates were due to viscosity *per se*, we used liquids of different kinematic viscosities and repeated the experiments at the larger amplitudes at which the amplitude-dependent growth rate appeared. An example of the amplitude-dependent resonance curves is shown in figure 5, and the effect of viscosity can be ascertained by comparing the curves for the liquids of different viscosity. Note there, also, that the small amplitude maximum amplification occurred around a nutational frequency of 0·056, whereas

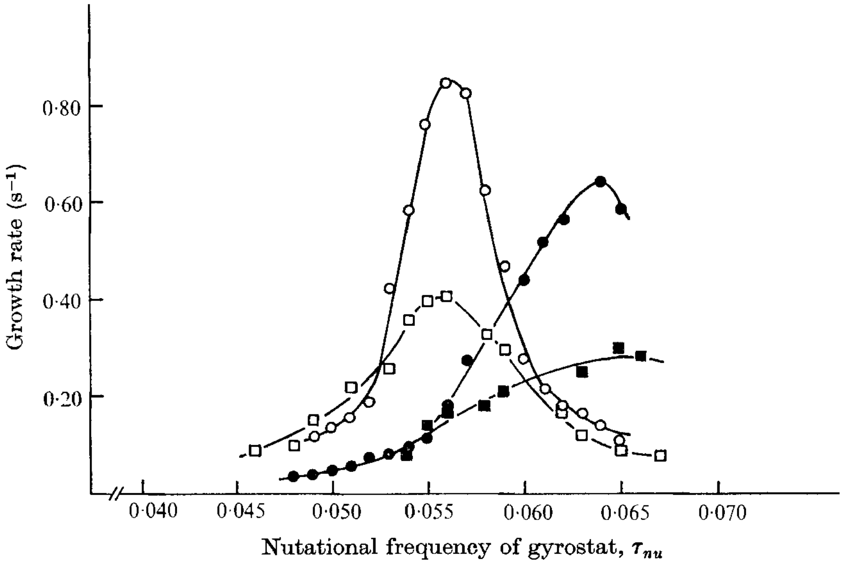

FIGURE 5. Resonance curves: amplitude-dependent (solid symbols) and amplitude-independent (open symbols) growth rates versus frequency. Cavity diameter = 2·50 in., cavity height = 7·48 in., angular speed = 5000 r.p.m., percentage of liquid = 79. ○, 1 centistoke oil; □, 13 centistoke oil.

for the larger amplitudes, maximum amplification occurred at a nutational frequency of about 0·064. Making the reasonable assumption that for large amplitude motion also $\tau_{nj} = \tau_{nu}$ when maximum amplification occurred, then, since maximum undamping occurred at a different value of τ_{nu}, one concludes that the characteristic frequency of oscillation of the liquid at the larger amplitudes is different from what it was at the smaller amplitude. Hence, the frequencies are amplitude dependent. Furthermore, from the qualitative similarity of the two curves for the liquids of different kinematic viscosity, it follows that viscosity *per se* is not the cause of the amplitude-dependent growth rate (the incompleteness of the amplitude-dependent curve is due to the restriction of the amplitude of the motion of the gyrostat to around 8°, this restriction being due partially to physical constraints and partially to the necessity of precluding possible resonance with any of the mechanical vibrations of the supposedly rigid supports).

The above procedure for generating the curves shown in figure 5 is a time-consuming one, for varying τ_{nu} by changing the brass rings on the rotor is tedious. An alternative procedure for obtaining resonance simply involves changing the amount of liquid in the cavity. This can be done while the gyroscope is spinning. Since the natural frequencies of the liquid depend on its geometry, i.e. the percentage present, this adjustment of the volume changes these eigenfrequencies without measurably altering the nutational frequency τ_{nu}. Thus, keeping τ_{nu} constant while adjusting the amount of liquid, one can easily shift the frequencies of the liquid closer to the nutational frequency. This procedure has two distinct advantages over obtaining resonance by altering the moments of inertia to

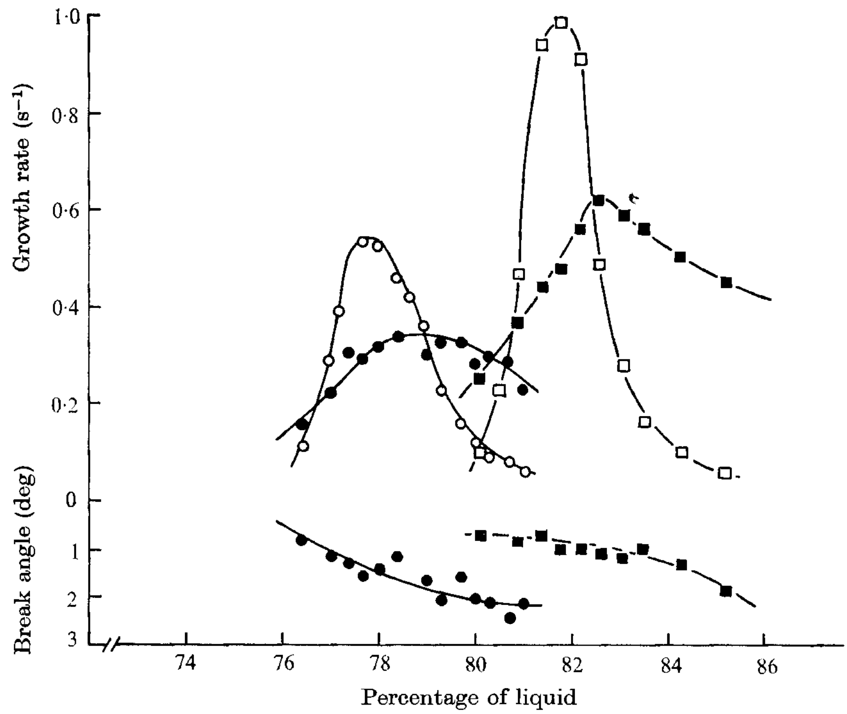

FIGURE 6. Amplitude-dependent (solid symbols) and amplitude-independent (open symbols) growth rates and 'break angles' versus percentage of liquid. Cavity diameter = 2·50 in., cavity height = 7·48 in., angular speed = 5000 r.p.m., nutational frequency τ_{nu} = 0·056. □, water; ○, 5 centistoke oil.

adjust τ_{nu}: (i) it is a simple operation; (ii) it leads to results that answer the question whether or not the amplitude-dependent behaviour of the growth rate is caused solely by the free surface.

For similar amplitude levels prescribed by previous tests wherein we obtained resonance by varying τ_{nu}, we show in figure 6 the resonance curves obtained by adjusting the volume of liquid. Also included in figure 6 are the 'break angles' (those nebulous values of the amplitude of the gyroscope below which the growth rate does not depend on the amplitude, but above which the growth rate does depend on the amplitude). Note that the resonance curves from these experiments are qualitatively similar to the curves in figure 5 in that the eigenfrequencies and amplification rates at the large amplitudes are different from those at the smaller amplitudes (the repeatability of the experiments, along with the complete absence of any hysteresis, was most reassuring).

In figure 6, the displacement between the two resonance peaks for liquids of different viscosity is consistent with the effect of viscosity (Karpov 1965), i.e. viscosity increases the resonant frequencies (which corresponds to decreasing the percentage of liquid) and depresses the amplitude of the gyroscopic motion (at resonance) from that for a liquid of lower viscosity. Note here, also, from the 'break angle' plot, that the region of overlap around the 82 % fill value seems to indicate that an increase in viscosity delays the onset of the amplitude-dependent

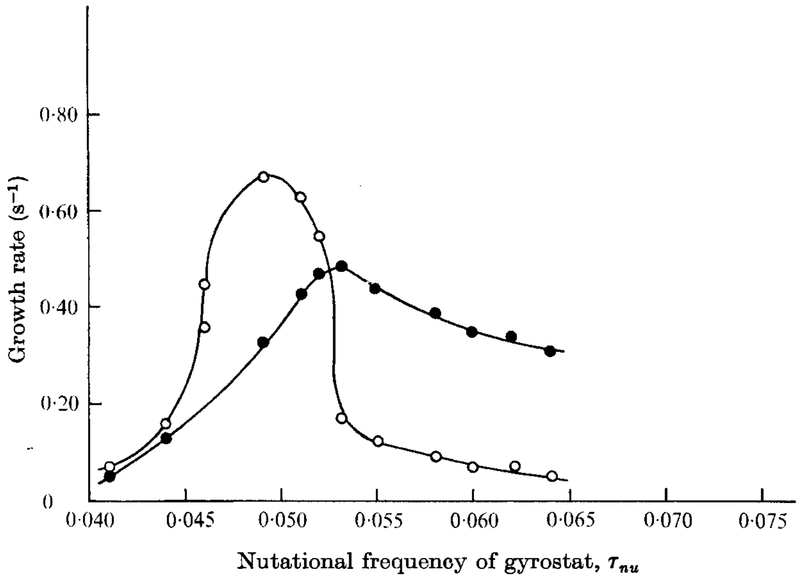

FIGURE 7. Amplitude-dependent (solid symbols) and amplitude-independent (open symbols) growth rates versus nutational frequency. Cylinder diameter = 2·48 in., cylinder height = 7·817 in., angular speed = 5000 r.p.m., cylinder completely filled with 1 centistoke oil.

response; that is, the 'break angle' for water (kinematic viscosity = 0·01 stokes) is about 1° whereas the 'break angle' for oil (kinematic viscosity = 0·05 stokes) is about 2°. We shall return to this important fact in the discussion.

That the amplitude-dependent response is not caused solely by the action of the free surface in a partially filled, uniformly rotating cylinder is revealed by comparing figures 5–7. Figure 7 shows the amplitude-independent and amplitude-dependent response curves for our gyrostat with a completely filled cylindrical cavity. The curves were generated by varying τ_{nu}. Figures 5 and 6 are for partially filled cavities and were generated by varying τ_{nu} and τ_{nj} respectively. The fact that all three figures exhibit the amplitude-dependent response seems to indicate that something more basic than the relative freedom of motion of a free surface is causing the effect.

3. Discussion

The Stewartson analysis does correctly predict the eigenfrequencies and amplification rates for our gyrostat when its amplitude of motion is no more than about 1°. Hence, we search for the reason his analysis fails at larger amplitudes by examining the simplifications he made on the governing partial differential equation, the Navier–Stokes equation.† He lowered the order of the equation by neglecting the viscous term, linearized it by neglecting the nonlinear term, and simplified the analysis by assuming that the axial spin $\mathbf{\Omega}$ of the liquid always

† This phenomenon is not just a large amplitude effect due solely to the gyrostat, for it is not observed if one operates the gyrostat at these large amplitudes either while it is empty or while it is filled with a solid of the same density as the liquid.

remained collinear with the axial spin of the gyrostat. Since we have already dismissed viscosity as a cause, it follows, then, that the amplitude-dependent response must be associated with the nonlinear convective term in the Navier–Stokes equation and/or the non-collinearity of the two spin vectors. It would not seem an unreasonable effort to examine the dynamic effect of the interaction, via this nonlinear term,† of two different modes of the inertial wave frequency spectrum, particularly the Rossby-type mode that can arise from the non-collinearity‡ of the two axial spin vectors mentioned above. If this were the mechanism for the amplitude-dependent response, then the previously mentioned effect of viscosity in delaying the amplitude-dependent response to larger angles could be understood in terms of viscosity tending to keep the two spin vectors aligned for longer.

The significant result of the investigation, from a utilitarian standpoint, comes from an examination of figures 5–7. If one is concerned about dynamic stability problems for a spinning liquid-filled body executing gyroscopic motion, and the amount of liquid can be varied, then, to avoid the spurious resonance regime, one should use an amount of liquid less than the amount that would be in a 'Stewartson' resonance at small amplitudes. If the amount of liquid cannot be changed and one wishes to avoid the spurious large amplitude response by changing τ_{nu}, one should change the design of the body so that it has a nutational frequency lower than the 'Stewartson' resonant frequency of the liquid.

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† There is a fluid dynamical instability theory of nonlinear oscillations (Stuart 1971) that predicts, for the fundamental mode of the oscillations, that the amplitude A of the oscillations is time dependent and satisfies $d|A|^2/dt = \alpha_1|A|^2+\alpha_2|A|^4$, where α_1 and α_2 are constants. This equation, however, does not allow for, or describe, wave propagation, whereas the effect we are observing (the growth in amplitude of the gyroscopic motion of the rigid body containing the liquid) is due to the interaction between the inertial waves in the liquid and the oscillatory motion of the gyrostat. This difference may or may not explain our inability to correlate Stuart's result with our experiments.

‡ Greenspan (1968, p. 86) shows that Rossby waves can occur in a sliced-off cylinder. If the spin vector of the liquid in the cavity is not always aligned with the spin vector of the rotor, the liquid sees, effectively, a sliced-off cylinder, and Rossby waves should coexist with the ordinary inertial waves.